Exact equations and scaling relations for $\overline{f_0}$ avalanche in the Bak-Sneppen evolution model

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An infinite hierarchy of exact equations is derived for the newly observed $\overline{f_0}$ avalanche in the Bak-Sneppen model. By solving the first-order exact equation, we find that the critical exponent γ, governing the divergence of the average avalanche size, is exactly 1 (for all dimensions), which has been confirmed by extensive simulations. Solution of the gap equation yields another universal result $\rho=1$ (ρ is the exponent of relaxation to attractor). Scaling relations are established among the critical exponents (γ, τ, D, σ, and ν) for the $\overline{f_0}$ avalanche.

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In the Bak-Sneppen (BS) evolution model [1], random numbers f_i, chosen from a flat distribution between 0 and 1, $p(f)$, are assigned independently to each species located on a d-dimensional lattice of linear size L. At each time step, the extremal site, i.e., the species with the smallest random number, and its $2d$ nearest neighboring sites, are assigned $2d+1$ new random numbers also chosen from $p(f)$. This updating continues indefinitely. After a long transient process the system reaches a statistically stationary state where the density of random numbers in the system is uniform above f_c (the self-organized threshold), and vanishes for $f<f_c$.

Despite the fact that it is an oversimplification of a real biological process, the BS model exhibits such common interesting features observed by paleontologists [2,3] as punctuated equilibria, power-law probability distributions of lifetimes of species, and sizes of extinction events. These behaviors suggest that the ecology of interacting species might have evolved to a self-organized critical state.

The BS model displays spatial-temporal complexity, which also emerges from many natural phenomena, such as fractals [4], $1/f$ noise [5], etc. This strongly suggests that various complex behaviors may be attributed to a common underlying mechanism. The authors of Ref. [6] suggested that the relation of these different phenomena can be established on the basis of their unique models. It was even proposed by them that spatial-temporal complexity comes out as a direct result of avalanche dynamics in driven systems, and different complex phenomena are related via scaling relations to the fractal properties of the avalanches. It can hence be inferred that avalanche dynamics plays a key role in dealing with complex systems, especially when one needs to know the macroscopic features of the systems, since lingering on the inner structure of individuals will not be helpful [7].

An avalanche is a kind of macroscopic phenomenon driven by local interactions. The size of an avalanche may be extremely sensitive to the initial configuration of the system, while the distribution of the sizes (spatial and temporal) of avalanches, i.e., the "fingerprint," should be robust with respect to the modifications, due to the universality of complexity and the definition of self-organized criticality (SOC) [8]. In this sense, the extent of what we know about an avalanche will determine to what extent we know a complex system. Avalanche dynamics provides insight into complexity, and enables one to further investigate the system studied.

Though avalanche dynamics may be a possible underlying mechanism of complexity, the definitions of avalanches can be vastly different for various complex systems, or for same sorts of systems, even for the same one. In the BTW model [9], an avalanche is caused by the adding of a grain or several grains of sand into the system. The avalanche is considered when the heights of all the sites are less than critical value, say, 4. In the BS model [1,6], several types of avalanches, for instance, the f_0 avalanche, the $G(s)$ avalanche, the forward avalanche, backward avalanche, etc., are presented. These different definitions of avalanches may show unique hierarchal structures, while they manifest common fractal feature of the complex system, that is, SOC. It can be inferred that various types of avalanches are equivalent in the sense that they imply complexity.

Since similar structures and common features evidently arise in different types of avalanches, it is straightforward that various avalanches differ from each other only in the contexts from which one comprehends them. As is known, the major aim of avalanche study is to investigate the universal rules possibly hidden behind the evolution of the systems or the models. Hence the means of understanding the avalanches appears crucial. Better ways may enable one to know more about the system or the model, and hence to have a better comprehension of the features corresponding to complexity. From this point of view, when studying avalanches one should try to choose easier ways instead of more difficult ones.

The evolution of the highly sophiscated BS model shows a hierarchal structure specified by avalanches, which correspond to sequential mutations below a certain threshold. It has been noted [10] that in the BS model an avalanche is initiated when the fitness of the globally extremal site (the

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species with the least random number) is larger than the self-organized threshold. That is, the triggering event of an avalanche is directly related to the fitness, the feature of individuals. In other words, the avalanche is directly associated with the feature of individuals instead of general features of the ecosystem as a whole. Is it feasible that the avalanches are directly caused by the global feature of the whole system? Can such a global feature be expressed in terms of the corresponding quantity? If such a quantity is found and such avalanches observed, may the new avalanches provide a new and easier way to investigate properties of the model?

One of our previous works [10] presented such a different hierarchy of avalanches (the $\overline{f}_0$ avalanche) for the BS model. We defined a global quantity $\overline{f}$ which denotes the average fitness of the system. The new avalanches are directly related to $\overline{f}$. In this paper, we present a master equation for the hierarchal structure of $\overline{f}_0$ avalanches. It prescribes the cascade process of smaller avalanches merging into larger avalanches when the critical parameter $\overline{f}_0$ is changed. An infinite series of exact equations can be derived from this master equation. The first order exact equation, together with a scaling ansatz of the average sizes of avalanches, shows the exact result of γ, the critical exponent governing self-organization, to be universally 1 for all dimensional BS models, which has been confirmed by extensive simulations of the model. We also establish scaling relations related to some critical exponents for the $\overline{f}_0$ avalanche, and make predictions on the values of some exponents.

The quantity $\overline{f}$ is a global one of the ecosystem and can be expected to involve some general information about the whole system. It may represent the average population or living capability of the whole species system. A larger $\overline{f}$ shows that the average population is immense or the average living capability is great, and vice versa. $\overline{f}$ is defined as

$$\overline{f}=\frac{1}{L^d}\sum_{i=1}^{L^d} f_i, \tag{1}$$

where f_i is the fitness of the ith species of a system consisting of L^d species. Let the BS model start to evolve. At each time step of the evolution, apart from the random numbers of the globally extremal site and its $2d$ nearest neighboring sites, the signal $\overline{f}$ is also tracked. Initially, $\overline{f}$ tends to increase stepwisely. As the evolution continues further, $\overline{f}$ approaches a critical value $\overline{f}_c$ and remain statistically stable around $\overline{f}_c$. The plot of $\overline{f}$ versus time step s shows that the increasing signals of $\overline{f}$ follow a Devil's staircase [8], which implies that a punctuated equilibrium emerges. Denote $F(s)$ as the gap of the punctuated equilibrium. Actually, $F(s)$ tracks the peaks in $\overline{f}$. After some careful derivation one can write an exact gap equation [6,10]

$$\frac{dF(s)}{ds}=\frac{\overline{f}_c-F(s)}{L^d\langle S\rangle_{F(s)}}, \tag{2}$$

where $\langle S\rangle_{F(s)}$ denotes the average size of avalanches that occurr during the gap $F(s)$ when $\overline{f}<F(s)$. This exact gap equation will be exactly solved in this paper.

Signals $\overline{f}(s)$ play important roles in defining the $\overline{f}_0$ avalanche. For any value of the auxiliary parameter $\overline{f}_0$ ($0.5<\overline{f}_0<1.0$), an $\overline{f}_0$ avalanche of size S is defined as a sequence of $S-1$ successive events when $\overline{f}(s)<\overline{f}_0$ confined between two events when $\overline{f}(s)>\overline{f}_0$. This definition ensures that the mutation events during an avalanche are spatially and temporally correlated. It can also guarantee the hierarchal structure of the avalanches: larger avalanches consist of smaller ones. As $\overline{f}_0$ is raised, smaller avalanches gather together and form larger ones. The statistics of $\overline{f}_0$ will inevitably have a cutoff if $\overline{f}_0$ is not chosen to be $\overline{f}_c$. This will not affect the size distribution, provided $\overline{f}_0$ approaches $\overline{f}_c$. Extensive simulations show that exponents τ of $\overline{f}_0$-avalanche size distribution are 1.800 and 1.725 for one-dimensional (1D) and 2D BS models, respectively, amazingly different from the counterparts of the f_0 avalanche, 1.07 and 1.245 [6]. This strengthens the speculation that the $\overline{f}_0$ avalanche is a different type of avalanche, distinguished from any types of avalanches found previously.

Denote by $P(S,\overline{f}_0)$ the probability of acquiring an $\overline{f}_0$ avalanche of size S. The signals $\overline{f}(s)$ $[\overline{f}_0<\overline{f}(s)<\overline{f}_0+d\overline{f}_0]$ will stop $\overline{f}_0$ avalanches and not $(\overline{f}_0+d\overline{f}_0)$ avalanches. That is, as $\overline{f}_0$ is raised by an infinitesimal amount $d\overline{f}_0$, some $\overline{f}_0$ avalanches merge together to form larger $(\overline{f}_0+d\overline{f}_0)$ avalanches. This exhibits the hierarchal structure of $\overline{f}_0$ avalanches, and will be prescribed by the exact master equation below. In some sense, the master equation reflects the ''flow'' of probability of avalanche size distribution with respect to the change in $\overline{f}_0$.

Simulations show that $\overline{f}$ approaches $\overline{f}_c$, and remains statistically stable in the critical state. This feature is greatly different from the feature of $f_{\rm min}$ (the fitness of the globally extremal site), which can vary between 0 and 1. $\overline{f}$ in the critical state fluctuates slightly around $\overline{f}_c$. Therefore, the $\overline{f}_0$ avalanches will have no good statistics if $\overline{f}_0$ is chosen as a value far less than $\overline{f}_c$, since there only exist smaller avalanches in the model. To acquire a better and reasonable distribution of $\overline{f}_0$-avalanche sizes, one should choose a value of $\overline{f}_0$ under the condition $\overline{f}_0\to\overline{f}_c$. It should be emphasized that the master equation listed below is also valid for $\overline{f}_0\to\overline{f}_c$.

Both theoretical analysis and extensive simulations suggest that the signals $\overline{f}(s)$ which terminate $\overline{f}_0$ avalanches are uncorrelated and evenly distributed between $(\overline{f}_0,\overline{f}_c)$, provided that $\overline{f}_0\to\overline{f}_c$. The direct consequence of this observation is that the probability of an $\overline{f}_0$ avalanche merging to $(\overline{f}_0+d\overline{f}_0)$ avalanche is prescribed by $d\overline{f}_0/(\overline{f}_c-\overline{f}_0)$. It is important to note that any two subsequent avalanches must be mutually independent for the following arguments to be true. In other words, the probability distribution of $\overline{f}_0$ avalanches, initiated immediately after the termination of an $\overline{f}_0$ avalanche of size S, must be independent of S. This is true because in the BS model the dynamics within an $\overline{f_0}$ avalanche is completely independent of the particular value of the signals $\overline{f(s)}>\overline{f_0}$ in the background that were left by the previous avalanches.

Here we present the master equation. As $\overline{f_0}$ is raised by an infinitesimal amount $d\overline{f_0}$, the probability ''flowing'' out of the size distribution of $\overline{f_0}$ avalanches is given by $P(S,\overline{f_0})[d\overline{f_0}/(\overline{f_c}-\overline{f_0})]$, while the probability ''flowing'' in is given by $\Sigma_{S_1=1}^{S-1}[P(S_1,\overline{f_0})/(\overline{f_c}-\overline{f_0})]P(S-S_1,\overline{f_0})$. If $\overline{f_0}\rightarrow\overline{f_c}$ and $d\overline{f_0}\rightarrow 0$, one can write the master equation as

$$(\overline{f_c}-\overline{f_0})\frac{\partial P(S,\overline{f_0})}{\partial\overline{f_0}} = -P(S,\overline{f_0})+\sum_{S_1=1}^{S-1}P(S_1,\overline{f_0})P(S-S_1,\overline{f_0}). \quad (3)$$

The first term on the right hand side of the equation expresses the loss of avalanches of size S due to the merging with the subsequent one, while the second one describes the gain in $P(S,\overline{f_0})$ due to merging of avalanches of size S_1 with avalanches of size $S-S_1$.

In order to investigate the exact master equation it is convenient to make some variable changes. Define $h=-\ln(\overline{f_c}-\overline{f_0})$. Therefore, $\overline{f_0}=\overline{f_c}$ corresponds to $h=+\infty$. Since in the master equation $\overline{f_0}$ is chosen to be close to $\overline{f_c}$, h varies from a very large number to $+\infty$. Due to the variable change the variable h is chosen from the distribution $P(h)=e^{-h}$, which seems to be more ''natural.'' In what follows we will use the new variable h instead of $\overline{f_0}$. The master equation can be rewritten, in terms of h, as

$$\frac{\partial P(S,h)}{\partial h}=-P(S,h)+\sum_{S_1=1}^{S-1}P(S,h)P(S-S_1,h). \quad (4)$$

Making a Laplace transformation of Eq. (4), after some calculation, one obtains

$$\frac{\partial\ln[1-p(\beta,h)]}{\partial h}=p(\beta,h), \quad (5)$$

where $p(\beta,h)=\Sigma_{S=1}^{\infty}P(S,h)e^{-\beta S}$. This exact equation is the key to this work. Many interesting physical features can be derived from it. As $h<+\infty$, the avalanche size will have a cutoff. The normalization of $P(S,h)$ can be expressed as $p(0,h)=\Sigma_{S=1}^{\infty}P(S,h)=1$. Expanding both sides of Eq. (5) as Taylor series throughout a neighborhood of the point $\beta=0$, one can immediately obtain

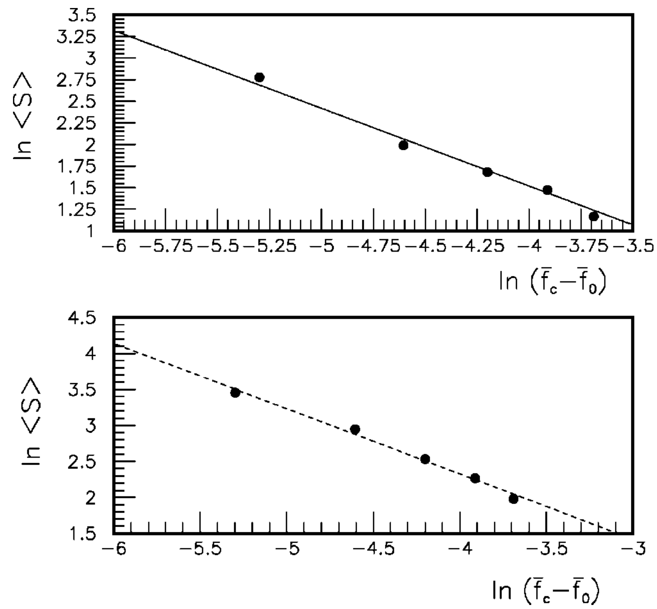

FIG. 1. The average size of avalanches $\langle S\rangle$ vs $(\overline{f_c}-\overline{f_0})$ for (a) 1D and (b) 2D Bak-Sneppen evolution models. The asymptotic slope yields $\gamma=0.99\pm0.01$ and 0.98 ± 0.01, respectively.

$$\frac{\partial}{\partial h}\left[1-\langle S\rangle_h\beta+\frac{1}{2!}\langle S^2\rangle_h\beta^2-\frac{1}{3!}\langle S^3\rangle_h\beta^3+\cdots\right] =\left[\langle S\rangle_h\beta-\frac{1}{2!}\langle S^2\rangle_h\beta^2+\frac{1}{3!}\langle S^3\rangle_h\beta^3+\cdots\right] \times\left[-1+\langle S\rangle_h\beta-\frac{1}{2!}\langle S^2\rangle_h\beta^2+\frac{1}{3!}\langle S^3\rangle_h\beta^3+\cdots\right]. \quad (6)$$

Since Eq. (6) holds for arbitrary β, comparing the coefficients of different powers of β in the above Taylor series gives an infinite series of exact equations. Comparison of the coefficients of β^1 results in

$$\frac{\partial\ln\langle S\rangle_h}{\partial h}=1. \quad (7)$$

Equation (7) is extremely interesting. Changing variable h back into $\overline{f_0}$, one can obtain the ''gamma'' equation [6,11]

$$\frac{d\ln\langle S\rangle_{\overline{f_0}}}{d\overline{f_0}}=\frac{1}{\overline{f_c}-\overline{f_0}}. \quad (8)$$

Inserting the scaling ansatz [6] $\langle S\rangle_{\overline{f_0}}\sim(\overline{f_c}-\overline{f_0})^{-\gamma}$ into Eq. (8), one immediately obtains an interesting result

$$\gamma=1. \quad (9)$$

It should be noted that $\gamma=1$ is universal, that is, independent of the dimension. The value of γ for $\overline{f_0}$ avalanches is different from those for the f_0 avalanche found in Ref. [6], which are 2.70 and 1.70 for 1D and 2D BS models, respectively. Extensive simulations show $\gamma=0.99\pm0.01$ and $\gamma=0.98\pm0.01$ for 1D and 2D BS models, respectively. Figure 1 shows our simulation results, which confirms the universal result $\gamma=1$.

Higher powers of β give new exact equations. Here we present the first two:

$$\frac{\partial}{\partial h}\left(\frac{\langle S^2\rangle_h}{\langle S\rangle_h}\right)=2\langle S\rangle_h, \tag{10}$$

$$\frac{\partial}{\partial h}\left(\frac{\langle S^3\rangle_h}{3\langle S\rangle_h}-\frac{\langle S^2\rangle_h^2}{2\langle S\rangle_h^2}\right)=\langle S^2\rangle_h. \tag{11}$$

Next we present the solution of the exact gap equation for $\overline{f_0}$ avalanches. Inserting the scaling relation $\langle S\rangle_{F(s)}\sim[\overline{f_c}-F(s)]^{-1}$ into the equation and integrating, one obtains

$$\Delta\overline{f}(s)=\overline{f_c}-F(s)\sim\left(\frac{s}{L^d}\right)^{-\rho}=\left(\frac{s}{L^d}\right)^{-1}, \tag{12}$$

where ρ is the exponent of the relaxation to the attractor [6]. Thus we obtain $\rho=1$. Interestingly, ρ is also a universal exponent for all dimensional BS models. It shows that the critical point $(\Delta\overline{f}=0)$ is approached algebraically with an exponent -1.

Up to now, we have obtained some exponents of corresponding physical properties of $\overline{f_0}$ avalanches: τ, the avalanche size distribution [10]; D, the avalanche dimension [10]; γ, the average avalanche size [10]; and ρ, the relaxation to the attractor [6]. Recall another two exponents [6] ν and σ, which are defined as $r_{co}\sim(\overline{f_c}-\overline{f_0})^{-\nu}$ and $S_{co}=(\overline{f_c}-\overline{f_0})^{-1/\sigma}$, respectively. Here r_{co} and S_{co} are referred to as the cutoff of the spatial extent of avalanches (due to the limit system size) and that of the avalanche size (due to the fact that $\overline{f_0}$ is not chosen as $\overline{f_c}$), respectively. It is natural to establish some scaling relations of these exponents for $\overline{f_0}$ avalanches similar to those found in Refs. [6,12] for f_0 avalanches. Nevertheless, these two types of avalanches manifest similar fractal properties. Hence some common features should be shared by them. Integrating the equation $\langle S\rangle=\int_1^{(\overline{f_c}-\overline{f_0})^{-(1/\sigma)}}SP(S,\overline{f_0})dS$ and the scaling $\langle S\rangle\sim(\overline{f_c}-\overline{f_0})^{-1}$ results in

$$\gamma=\frac{2-\tau}{\sigma}=1. \tag{13}$$

Due to the compactness [6] of avalanches, we have $S_{co}\sim r_{co}^D=(\Delta f)^{-\nu D}$; thus

$$\nu=\frac{1}{\sigma D}=\frac{1}{(2-\tau)D}. \tag{14}$$

Equations (11) and (12) establish scaling relations among the critical exponents, and they imply that the self-organization time to reach the critical state is independent of the initial configuration of the system. A system of size L reaches the stationary state when $[\Delta f(s)]^{-\nu}\sim L$. It can be inferred from Eqs. (11) and (12) that, if one chooses τ and D as two independent exponents, other exponents can be expressed in terms of them. Among the six exponents mentioned above, τ and D can be numerically measured [10], and γ and ρ can be analytically obtained, while ν and σ are difficult to explore despite the fact that some methods measuring the corresponding exponents for f_0 avalanches were introduced in Ref. [13]. Therefore, we can rely on the scaling relations and values of the exponents obtained to predict the values of ν and σ. We predict $\sigma=0.2$ (one dimension) and 0.275 (two dimensions), $\nu=2.04$ (one dimension) and 1.17 (two dimensions).

Comparing the $\overline{f_0}$ avalanche with the f_0 avalanche, we find that the former is more readily treated. Two critical exponents can be analytically obtained, and are found to be universal for all dimensional BS models. Furthermore, the infinite hierarchy of exact equations and the exact gap equations, together with their solutions, provide an exclusive investigation of the new type of avalanche. Another asset of the $\overline{f_0}$ avalanche is that it involves some information concerning the whole system. It can be concluded that the $\overline{f_0}$ avalanche does enable us to comprehend the complex system from an effective and different context. The weak point of this avalanche is that it loses some knowledge of individuals. It is still unknown how these individual features will matter. It is worthwhile to investigate the avalanche dynamics further in the future.

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